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Smoluchowski's Coagulation Equation: Probabilistic Interpretation of Solutions for Constant, Additive and Multiplicative Kernels

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Abstract. This paper is devoted to the study of the Smoluchowski's coagulation equation, discrete and continuous version, for the case of constant, additive and multiplicative kernels. Even though, for the discrete case the results stated in this work are not new, our approach allows the simplification of existing proofs. For the continuous case we obtain new results: a connection between the solutions of the additive and multiplicative cases and renormalisation theorems which show that after a convenient scaling, the solution converges to a limit which depends on the initial condition only through its moments of order 1, 2 and 3.

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1. – Introduction

1.1. – Paper's plan

The aim of this paper is to provide a probabilistic representation for some solutions of the Smoluchowski's coagulation equation.

In the introduction we give the heuristic motivation of this problem. Afterwards we furnish a survey of some results that we can find in the literature on the Smoluchowski's equation, without any intention of being exhaustive, but by pointing out the results obtained by using probabilistic methods. For a more detailed survey we refer to Aldous ([Ald99]).

The two following parts discuss three particular kernels: constant, additive and multiplicative. This choice for the kernel allows the development of the computation.

The second part recalls briefly some known results on the discrete case for this three particular kernels. We obtain the explicit form of these solutions via branching processes. The results we give on this part are not new but our

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approach allows the simplification of existing proofs.

The last part deals with the continuous case. We express here our main results: a transformation which connects the solution of the additive case with the one of the multiplicative case (Theorems 3.9 and 3.11), and some renormalisation theorems (Theorems 3.6, 3.20 and 3.24). These last theorems insure the convergence of the solution to a limit, which depends weakly on the initial condition.

1.2. – Heuristic motivation of the problem

The Smoluchowski's coagulation equation models various kind of phenomena as for example: in chemistry (polymerisation), in physics (aggregation of colloidal particles), in astrophysics (formation of stars and planets), in engineering (behaviour of fuel mixtures in engines), in genetics, in random graphs theory etc.

In order to fix the ideas, we present the appearance of this equation, in polymerisation.

For $k \in \mathbb{N}^*$, let P_k denote a polymer of mass k, that is a set of k identical particles (monomers). As time advances, the polymers evolve and, if they are sufficiently close, there is some chance that they merge into a single polymer whose mass equals the sum of the two polymers' masses which take part in this binary reaction. By convention, we admit only binary reactions. This phenomenon is called *coalescence* and we write formally

$$P_k + P_j \longrightarrow P_{k+j} ,$$

for the coalescence of a polymer of mass k with a polymer of mass j.

Let $n(k,t)$ denote the average number of polymers of mass k per unit volume, at time t. The expression $k\,n(k,t)$ denotes the part of mass consisting on polymers of length k, per unit volume.

It is thus natural to consider that the coalescence phenomenon ($P_k + P_j \longrightarrow P_{k+j}$), is proportional to $n(k,t)\,n(j,t)$ with a proportionality constant $K(k,j)$, called coalescence kernel.

In the sequel we employ for the discrete case letters i, j, k... while for the continuous case we use x, y, z.... Furthermore, throughout this paper, time t is always continuous, discrete and continuous refer to polymers' masses.

Hereafter (discrete and continuous case), the coagulation kernel K will satisfy the following hypothesis:

(H_1) K is positive i.e., $K : (\mathbb{N}^*)^2$ or $(\mathbb{R}_+)^2 \to \mathbb{R}_+$,

and

(H_2) K is symmetric i.e., $K(i,j) = K(j,i)$.

The Smoluchowski's coagulation equation, in the discrete case, is the equation on $n(k,t)$, for $k \in \mathbb{N}^*$. It is usually written on the following form

$$\text{(SD)} \quad \begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} K(j,k-j)n(j,t)n(k-j,t) - n(k,t)\sum_{j=1}^{\infty} K(j,k)n(j,t) \\ n(k,0) = n_0(k), \ k \geq 1. \end{cases}$$

This system describes a non linear evolution equation of infinite dimension, with initial condition $(n_0(k))_{k\geq 1}$. Due to the presence of the infinite series, (SD) is not a classical initial value problem for a system of non linear ordinary differential equations, and even the existence of a local solution is not guaranteed by the theory of ordinary differential equations. According to the form of the coalescence kernel K and the one of the initial condition, we obtain or not solutions for this system. In the first line of (SD), the first term on the right hand side describes the creation of polymers of mass k by coagulation of polymers of mass j and $k-j$. The coefficient $\frac{1}{2}$ is due to the fact that K is symmetric. The second term corresponds to the depletion of polymers of mass k after coalescence with other polymers.

The notion of solution will be given in the following section (Definitions 1.1, 1.2 for the discrete case and Definitions 1.3, 1.4 for the continuous case).

The continuous analogue of the equation (SD) can be written naturally

$$\text{(SC)} \quad \begin{cases} \dfrac{\partial}{\partial t}n(x,t) = \dfrac{1}{2}\displaystyle\int_0^x K(y,x-y)n(y,t)n(x-y,t)dy \\ \qquad\qquad\qquad - n(x,t)\displaystyle\int_0^{\infty} K(x,y)n(y,t)dy \\ n(x,0) = n_0(x). \end{cases}$$

1.3. – Some known results

The references, while numerous, are not intended to be complete, except that we have sought to represent the major direction of research from a probabilistic point of view. In this study on the Smoluchowski's coagulation equation we consider three kernels: constant, additive and multiplicative.

1.3.1. – Analysis of the discrete case

Let us consider the discrete case. Introduce first of all the notion of weak solution for the system (SD) (we can find this definition for example in Laurençot ([Lau99])).

DEFINITION 1.1. Let $T \in (0,\infty]$ and $(n_0(k))_{k\geq 1}$ be a sequence of positive real numbers. We call *weak solution* of the system (SD) on $[0,T)$, a sequence of nonnegative continuous functions such that, for all $t \in (0,T)$ and $k \geq 1$:

(a) $n(k,.) \in \mathcal{C}([0,T])$ and $\sum_{j=1}^{\infty} K(j,k)\,n(j,t) \in L^1(0,t)$, and

(b)

$$n(k,t)=n_0(k)+\int_0^t\left(\frac{1}{2}\sum_{j=1}^{k-1}K(j,k-j)n(j,s)n(k-j,s)-n(k,s)\sum_{j=1}^{\infty}K(j,k)n(j,s)\right)ds.$$

For our study we shall consider rather the notion of strong solution.

DEFINITION 1.2. Let $T \in (0,\infty]$ and $(n_0(k))_{k\geq 1}$ be a sequence of positive real numbers. We call *strong solution* of (SD) on $[0,T)$, a sequence of positive functions such that, for all $t \in (0,T)$ and $k \geq 1$: the derivative of $n(k,t)$ with respect to t exists, $\sum_{j=1}^{\infty} K(j,k)\,n(j,t) < \infty$ and (SD) is verified.

A *global solution* is a solution with $T=\infty$. The solution will be called *local* in the opposite case.

Hereafter, we denote by C a constant whose value changes from line to line.

Let us make the following remark. Formal computations give

$$\frac{d}{dt}\sum_{k=1}^{\infty} k\,n(k,t) = 0. \tag{1.1}$$

Therefore it is natural to ask if (1.1) is preserved on time. From a physical point of view the equality (1.1) is equivalent to the conservation of the mass for the system (SD). As far as (1.1) is true we have

$$\sum_{k=1}^{\infty} k\,n(k,t) = \sum_{k=1}^{\infty} k\,n(k,0). \tag{1.2}$$

The first moment for which (1.2) is not longer valid is called *gelification time* and is defined by

$$T_{\mathrm{gel}} = \inf\left\{ t \geq 0;\ \sum_{k=1}^{\infty} k\,n(k,t) < \sum_{k=1}^{\infty} k\,n(k,0)\right\}. \tag{1.3}$$

In polymerisation this time corresponds to the appearance of an infinite polymer called gel (and is equivalent to mass removal). From a physical point of view the gelation phenomenon might be interpreted as follows: the process of formation of large polymers takes place at sufficiently large rate so that a part of monomers is transfered to larger and larger polymers, and eventually gives rise to a huge polymer called gel. This gel can be regarded as a polymer formed by an infinite number of monomers so it is not accounted into (SD).

Let us introduce also, the space of real positive sequences defined by

$$E = \left\{ x = (x_i)_{i\geq 1},\ x_i \geq 0,\ \sum_{i=1}^{\infty} i\,x_i < \infty \right\}. \tag{1.4}$$

Constant Kernels

In 1916 Smoluchowski ([Smo16]) studied (SD) for the constant kernel and gave an explicit solution. More generally, Melzak ([Mel57]) proved that, for

$$K(i,j) \leq C, \quad \text{for} \quad i,j \geq 1 \quad \text{and} \quad C > 0, \tag{1.5}$$

one can obtain existence and uniqueness of a global solution for (SD), for any initial condition in E.

Additive Kernels

Ball and Carr ([BC90]) were interested on kernels satisfying

$$K(i,j) \leq C(i+j), \quad i,\ j \geq 1. \tag{1.6}$$

They got the existence of a global solution for (SD), which conserves the mass, for any initial condition from E. In order to get uniqueness of this solution we have either to impose

$$K(i,j) \leq C\sqrt{ij}, \tag{1.7}$$

or to restrict the class of initial conditions to E. This last situation was treated by Heilmann ([Hei92]). For kernels satisfying the inequality (1.6) and with initial condition $(n(k,0))_{k\geq 1}$, from E, such that

$$\sum_{k=1}^{\infty} k^2 n(k,0) < \infty, \tag{1.8}$$

Heilmann ([Hei92]) proved the existence and uniqueness of a global solution for (SD), which satisfies for all $T \in [0,+\infty)$

$$\sup_{t\in[0,T]} \sum_{k=1}^{\infty} k^2 n(k,t) < \infty. \tag{1.9}$$

Previously, the case $K(i,j) = C(i+j)$ was considered by Golovin ([Gol63]). For "additive" kernels K of the form

$$K(i,j) = i^\alpha + j^\alpha \tag{1.10}$$

with $\alpha > 1$, Carr and da Costa ([CdC92]) proved that there is no solution (even local) of (SD). This result holds for any initial condition in E, different from the 0 function.

Multiplicative Kernels

The case of the kernel $K(i,j) = i\,j$ under the initial condition $n(k,0) = \delta_1(k)$, for all $k \in \mathbb{N}^*$, was treated by McLeod ([McL62]). This initial condition insures that, at $t = 0$, there are only monomers. McLeod deduced the existence and uniqueness of a local solution for all $t \in [0, 1)$ verifying

$$\sup_{s \in (0,t]} \sum_{k=1}^{\infty} k^2 n(k, s) < \infty, \qquad \forall t \in (0, 1). \tag{1.11}$$

For the same kernel and under the same initial condition, Kokholm ([Kok88]) proved the existence of an unique global solution for (SD), satisfying for all $T \in [0, +\infty)$

$$\sup_{t \in [0,T]} \sum_{k=1}^{\infty} k n(k, t) < \infty. \tag{1.12}$$

This solution is given by

$$n(k, t) = \begin{cases} \frac{k^{k-3}}{(k-1)!} t^{k-1} e^{-kt} & \text{for } t \in [0, 1] \\ \frac{k^{k-3}}{(k-1)!} \frac{e^{-t}}{t} & \text{for } t \in [1, +\infty). \end{cases}$$

Leyvraz and Tschudi ([LT81]), obtained a more general result. More precisely, for

$$K(i, j) \geq Cij, \qquad \forall i, j \geq 1, \tag{1.13}$$

any solution of (SD), when it exists, cannot satisfy the mass conservation for all t. Furthermore, for the initial condition $n(k, 0) = \delta_1(k)$ they constructed a solution for which:

$$\sum_{k=1}^{\infty} k\, n(k, t) = \begin{cases} 1 & \text{for } t \leq 1 \\ \dfrac{1}{t} & \text{for } t \geq 1. \end{cases}$$

Obviously, in this particular case $T_{\text{gel}} = 1$. Their result explains also the local solution obtained previously by McLeod ([McL62]).

Furthermore, Leyvraz ([Ley84]) treated the case $K(i, j) = i^\alpha j^\alpha$, and proved that for all $\alpha \in (\frac{1}{2}, 1]$, there exists an initial condition such that $T_{\text{gel}} = 0$ (that is the total mass decreases from the beginning).

Several works in the literature were concentrated on kernels which combine the additive, constant and multiplicative kernels, of the following form:

$$K(i, j) = A + B(i + j) + Cij, \tag{1.14}$$

where A, B and C are given constants.

For these kernels, the existence of a solution for the Smoluchowski's coagulation equation was obtained by using various techniques: Flory ([Flo41a], [Flo41b], [Flo41c]) and Spouge ([Spo83b], [Spo83c]), from a combinatorial point of view, Gordon ([Gor62]) and Spouge ([Spo83a]), by using branching processes.

1.3.2. – Analysis of the continuous case

Let us introduce first the notion of weak solution in the continuous case.

DEFINITION 1.3. Let $T \in (0, \infty]$ and $(n_0(x))_{x \geq 0}$ be a positive real function. We call *weak solution* of (SC) on $[0, T)$, a set of continuous and positive functions satisfying, for all $t \in (0, T)$ and $x \geq 0$:

(a) $n(x, .) \in \mathcal{C}([0, T])$ and $\int_0^\infty K(x, y) n(y, t) dy \in L^1(0, t)$

and

(b)

$$n(x,t) = n_0(x) + \int_0^t \left(\frac{1}{2} \int_0^x K(y, x-y) n(y,s) n(x-y, s) dy - n(x,s) \int_0^\infty K(x,y) n(y,s) dy \right) ds.$$

The notion of strong solution becomes then:

DEFINITION 1.4. Let $T \in (0, \infty]$ and $(n(x, 0))_{x \geq 0}$ be a real positive function. We call *strong solution* of (SC) on $[0, T)$, a set of positive and continuous functions such that, for all $t \in (0, T)$ and $x \geq 0$: the derivative with respect to t of $n(x, t)$ exists, $\int_0^\infty K(x, y) n(y, t) dy < \infty$ and (SC) is verified.

For the additive, constant and multiplicative kernels, in the continuous case, the uniqueness is far of being obvious. We can obtain it for some special kernels, as for example, those satisfying ([Ald99])

$$K(x, y) \leq C(1 + x + y). \tag{1.15}$$

More precisely, if the condition (C_t)

$$(C_t) \quad \int_0^\infty n(x, t) dx < \infty, \quad \int_0^\infty x n(x, t) dx = 1 \text{ and } \int_0^\infty x^2 n(x, t) dx < \infty,$$

is satisfied for $t = 0$, then the solution of the continuous Smoluchowski's coagulation equation (SC), exists and is unique. Furthermore (C_t) is valid for all $t \in [0, \infty)$.

The work of Drake ([Dra72]) and Aldous ([Ald99]) gave examples of kernels K which have appeared in physics and chemistry. The model initially proposed by Smoluchowski ([Smo16]) in 1916 had a kernel of the form

$$K(x, y) = C(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{-\frac{1}{3}} + y^{-\frac{1}{3}}), \tag{1.16}$$

and corresponds to a coagulation controled by the Brownian diffusion.

Ernst, Ziff and Hendricks ([EZH84]) gave in their paper the construction of a solution for (SC) after the gelification time, for kernels admitting a finite gelification time, of the form

$$K(x, y) = (xy)^\alpha, \text{ with } \frac{1}{2} < \alpha \leq 1. \tag{1.17}$$

For kernels of the form

$$K(x, y) = A + B(x + y) + Cxy, \tag{1.18}$$

Spouge ([Spo84]) studied the "critical" time

$$t_c = \inf\{t; \int_0^\infty x^2 n(x, t)dx < \infty\}, \tag{1.19}$$

and proved that it corresponds to a critical branching process. Times $t > t_c$ correspond to super-critical branching processes.

1.3.3. – Probabilistic approach for the Smoluchowski's coagulation equation

Many authors have treated the Smoluchowski's coagulation equation by using probabilistic methods.

In his paper, Aldous ([Ald99]) made a survey of the present situation for (SD) and (SC) from a probabilistic point of view. He brought also to the fore some open problems which can be solved by using probabilistic methods.

Lang and Nguyen ([LN80]) used the propagation of chaos method in order to prove the convergence of an infinite particles system, directed by 3 dimensional Brownian Motions, to the initial model of coagulation proposed by Smoluchowski ([Smo16]).

Recently, Jeon ([Jeo98]) approached the solution of a more general equation than (SD), in that, we have also the fragmentation of polymers, by a sequence of finite Markov chains. Jeon gave a general result. More precisely, if we have

$$K(i, j) \geq (ij)^\alpha, \text{ with } \alpha \in \left(\frac{1}{2}, 1\right), \tag{1.20}$$

and furthermore

$$\lim_{i+j\to\infty} \frac{K(i, j)}{ij} = 0, \tag{1.21}$$

then we have gelification in finite time ($T_{\text{gel}} < \infty$), for a large class of initial conditions. This result was generalised to the continuous situation by Norris ([Nor99]).

The equation for the kernel $K(i, j) = ij$ can be connected to studies on coagulation via random graphs and forests ([Ald99], [EP98]).

Another interesting direction in the present study of the Smoluchowski's coagulation equation is the approximation of the solution by using Monte Carlo methods ([Gui98], [Bab99]).

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2. – Connection between the discrete Smoluchowski's coagulation equation (SD) and branching processes

Let us consider the discrete Smoluchowski's coagulation equation (SD). We are going to study three coagulation kernels K: constant, additive and multiplicative. The initial condition that we consider for these three cases is $n(k,0) = \delta_1(k)$. It corresponds to an initial configuration in which there are only monomers. In the sequel we shall use the notion of solution in the sense of strong solution, given in the Definition 1.2.

For each one of these cases we shall emphasise a connection between the solution $n(k,t)$ of (SD) and a branching process. This allows to obtain explicit solutions (Corollaries 2.3, 2.5 and 2.7). The choice of this initial condition is essential and it corresponds, for the branching process, to the fact that it has one ancestor. On one hand, our results are not new, we can find them for example in Aldous ([Ald99]). On the other hand, our approach is new and consists in using generating functions, which allow the simplification of existing proofs. The goal of this section is to exhibit clearly the breadth and importance of branching processes in the study of the discrete Smoluchowski's coagulation equation.

A formal calculation allows the following remark.

LEMMA 2.1. *Before the gelification time ($t < T_{\text{gel}}$), we have for* (SD)

$$\frac{d}{dt}\sum_{k=1}^{\infty} k\, n(k,t) = 0. \tag{2.1}$$

By re-normalising the initial condition, we can always suppose that

$$\sum_{k=1}^{\infty} k\, n(k,t) = 1. \tag{2.2}$$

2.1. – Discrete coagulation equation with $K(i,j) = i\,j$

For this particular case the equation (SD) can be written

$$(\text{SD}*)\quad \begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} j(k-j)n(j,t)n(k-j,t) - kn(k,t)\sum_{j=1}^{\infty} jn(j,t) \\ n(k,0) = \delta_1(k),\ k \geq 1. \end{cases}$$

The choice of this initial condition corresponds to consider that at $t = 0$ we have only monomers. It is essential for the results which follow. We have $T_{\text{gel}} = 1$ (cf. Leyvraz and Tschudi ([LT81])), and we shall consider only $t \leq T_{\text{gel}}$. By using Lemma 2.1, we have the conservation of mass

$$\sum_{k=1}^{\infty} k\, n(k,t) = \sum_{k=1}^{\infty} k\, n(k,0) = 1, \quad \forall t \leq T_{\text{gel}} = 1. \tag{2.3}$$

Let us denote by

$$p(k,t) = k\, n(k,t), \tag{2.4}$$

such that $(p(k,t))_{k \geq 1}$ form a probability distribution on $\mathbb{N}^*$. The generating function of this probability,

$$G(t,s) = \sum_{k=1}^{\infty} p(k,t)\, s^k, \quad |s| \leq 1, \tag{2.5}$$

verifies

PROPOSITION 2.2. *The function G, given by* (2.5), *is the unique solution of the non linear partial differential equation*

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,s) = \dfrac{s}{2}\dfrac{\partial G^2}{\partial s}(t,s) - s\dfrac{\partial G}{\partial s}(t,s) \\ G(t,1) = 1 \\ G(0,s) = s, \end{cases} \tag{2.6}$$

for all $t \leq 1 = T_{\text{gel}}$ and $|s| \leq 1$.

The proof of this proposition is obvious in view of the equation (SD$*$), on $n(k,t)$. The solution of the equation (2.6) is also the generating function of the total population of a particular branching process. Indeed, let us consider a branching process with one ancestor and offspring distribution a Poisson distribution of parameter t.

Denote by X_t its total population. We consider the situation $X_t < \infty$ which is equivalent with $t \leq 1$ (by using classical properties of branching processes). The generating function of X_t satisfies the equation (2.6). We remark that on one hand, from a probabilistic point of view, $t = 1$ is a critical time in that, for $t > 1$ the probability that the total population of the branching process be finite is strictly less than 1 (that is $P\{X_t < \infty\} < 1$). On the other hand, for the Smoluchowski's coagulation equation (SD$*$), this time is the gelification time. It becomes thus natural to consider for both cases $t \leq 1$.

The equality (via uniqueness), of these two generating functions allows us to express the solution of (SD$*$)

COIROLLARY 2.3. *For $t \leq 1$, the solution of the equation* (SD$*$), *is given by:*

$$n(k,t) = \frac{1}{k^2}\frac{(kt)^{k-1}}{(k-1)!}e^{-kt}, \quad \forall k \in \mathbb{N}^*. \tag{2.7}$$

PROOF. This result follows immediately from a remarkable property of branching processes, which connects the law of the total population to the law of the offspring (Athreya and Ney ([AN72])). More precisely, if X_t denotes the total population of the previous branching process, we have

$$P(X_t = k) = \frac{1}{k}P(Y_t^1 + \dots + Y_t^k = k-1),$$

where Y_t^i are i.i.d. random variables having a Poisson distribution of parameter t. Thus, we have obtained for the total population X_t, a Borel distribution of parameter t. □

The previous corollary implicitly gives an existence and uniqueness result for the solution of the equation (SD$*$). We observe that, the connection between the solution of (SD$*$) and the branching process with one ancestor and offspring distribution a Poisson distribution of parameter t, denoted by $\mathcal{P}(t)$, allows to find the explicit form for the solution of (SD$*$), by using a generating function argument.

2.2. – Discrete coagulation equation with $K(i,j) = i + j$

The form of the equation (SD) becomes in this case

$$\text{(SD+)} \quad \begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{k}{2}\displaystyle\sum_{j=1}^{k-1} n(j,t)n(k-j,t) - k\,n(k,t)\sum_{j=1}^{\infty} n(j,t) - n(k,t) \\ n(k,0) = \delta_1(k), \ k \geq 1. \end{cases}$$

We have now $T_{\text{gel}} = \infty$ and the solution will be defined on the real positive line. By summing over k in (SD+) we deduce

$$\sum_{k=1}^{\infty} n(k,t) = e^{-t}. \tag{2.8}$$

This leads us to choose the normalisation

$$p(k,t) = e^t n(k,t). \tag{2.9}$$

The generating function associated with $(p(k,t))_{k\geq 1}$

$$G(t,s) = \sum_{k=1}^{\infty} p(k,t)s^k, \quad |s| \leq 1, \tag{2.10}$$

satisfies

PROPOSITION 2.4. *The function $G(t,s)$, given by* (2.10), *is the unique solution of the following partial differential equation:*

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,s) = s\,e^{-t}(G(t,s)-1)\,\dfrac{\partial G}{\partial s}(t,s) \\ G(t,1) = 1 \\ G(0,s) = s, \end{cases} \tag{2.11}$$

for all $t \in [0,+\infty)$ and $|s| \le 1$.

We don't prove this proposition. It follows easily from the equation (SD+). Let us construct a branching process for which the generating function of its total population is G. In order to do this, consider a branching process with one ancestor and offspring distribution a Poisson distribution of parameter $1 - e^{-t}$, $t \ge 0$. The generating function of its total population X_t, is the unique solution of (2.11). Classical results on branching processes allow to express the distribution of X_t. We can deduce thus the solution of the equation (SD+):

COROLLARY 2.5. *The solution $n(k,t)$ of the equation* (SD+), *is given by*

$$n(k,t) = \frac{1}{k}\frac{(k(1-e^{-t}))^{k-1}}{(k-1)!}e^{-t}\,e^{-k(1-e^{-t})},\ k \ge 1,\ t \ge 0. \tag{2.12}$$

Let us also remark that the previous corollary furnishes an existence and uniqueness result for the solution of (SD+).

2.3. – Discrete coagulation equation with $K(i,j) = 1$

The equation (SD) can be written in this case

$$\begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} n(j,t)n(k-j,t) - n(k,t)\sum_{j=1}^{\infty} n(j,t) \\ n(k,0) = \delta_1(k),\ k \ge 1. \end{cases} \tag{SD1}$$

By summing over k in (SD1) we deduce

$$\sum_{k=1}^{\infty} n(k,t) = \frac{2}{t+2}.$$

This leads us to consider

$$p(k,t) = \left(1 + \frac{t}{2}\right) n(k,t), \tag{2.13}$$

such that $(p(k,t))_{k\ge1}$ form a probability distribution. The generating function associated with this probability:

$$G(t,s) = \sum_{k=1}^{\infty} p(k,t)s^k, \quad |s| \le 1, \tag{2.14}$$

satisfies the following equation:

PROPOSITIOIN 2.6. *The function G, given in* (2.14), *is the unique solution of the partial differential equation*

$$\begin{cases} (2+t)\dfrac{\partial G}{\partial t}(t,s) = G(t,s)(G(t,s)-1) \\ G(t,1) = 1 \\ G(0,s) = s, \end{cases} \tag{2.15}$$

for all $t \in [0,+\infty)$ *and* $|s| \le 1$.

Again we observe that the proof of this result is obvious in view of the equation (SD1). Associate with G a branching process. While we are on the subject, consider a branching process with one ancestor and offspring distribution a Bernoulli distribution of parameter $p_t = \frac{t}{2+t}$. Denote by X_t its total population. The generating function of X_t is solution of (2.15) and, by uniqueness, we are able to express $n(k,t)$ with respect to $P(X_t = k)$. This remark gives us the form of the solution of the Smoluchowski's coagulation equation for $K(i,j) = 1$.

COROLLARY 2.7. *The solution* $n(k,t)$ *of the equation* (SD1), *is given by*

$$n(k,t) = \left(1+\frac{t}{2}\right)^{-2}\left(\frac{t}{t+1}\right)^{k-1}, \quad k \ge 1, \; t \ge 0. \tag{2.16}$$

REMARK 2.8. In the three preceding sections, the choice of the initial condition $n(k,0) = \delta_1(k)$ is essential in order to be able to use properties of branching processes with one ancestor.

Let us denote by $\mathcal{P}(\lambda)$ a Poisson distribution of parameter λ and by $\mathcal{B}(\lambda)$ a Bernoulli distribution of parameter λ. We use the following scheme to summarise the results for the discrete case

$K(k,j)$	$n(k,t)$	T_{gel}	Offspring distribution
1	$\left(1+\frac{t}{2}\right)^{-2}\left(\frac{t}{t+1}\right)^{k-1}$	∞	$\mathcal{B}\left(\frac{t}{2+t}\right)$
$k+j$	$\frac{1}{k}\frac{(k(1-e^{-t}))^{k-1}}{(k-1)!}e^{-t}\,e^{-k(1-e^{-t})}$	∞	$\mathcal{P}(1-e^{-t})$
kj	$\frac{1}{k^2}\frac{(kt)^{k-1}}{(k-1)!}e^{-kt}$	1	$\mathcal{P}(t)$

REMARK 2.9. We have presented in this section a simple manner which allows to obtain a probabilistic representation of solutions for the equation (SD),

by making use of the PDEs verified by a generating function associated with (SD).

Our approach gives a construction for each fixed time t. We refer to the Aldous' paper ([Ald99]) for a dynamic construction in t, via processes for the constant, additive and multiplicative coalescence (see section 3, constructions 5, 6 and 8 of his paper). Jeon ([Jeo98]) gives also an interesting construction for a general coagulation-fragmentation equation as a limit of a sequence of finite state Markov chains.

3. – Continuous Smoluchowski's coagulation equation (SC)

We shall now be interested on the Smoluchowski's coagulation equation (SC), for which the mass is a continuous parameter. First of all we present some general results for (SC), which are independent from the kernel K. Part of these results (Proposition 3.1), are taken from Aldous ([Ald99]).

Afterwards, as for the discrete case we shall consider the constant, additive and multiplicative kernels.

The aim of this part is to present some new results, more precisely: a duality result connecting the solutions of the multiplicative case with those of the additive case (Theorems 3.9 and 3.11) and also some renormalisation theorems (Theorems 3.6, 3.20 and 3.24), which insure the convergence of the solution, for a large class of initial conditions, to a limit, which depends weakly on the initial condition.

3.1. – General results for (SC)

Hereafter we shall use for solution the notion of strong solution introduced in Definition 1.4.

PROPOSITION 3.1. *Let K be a symmetric and positive kernel and $n(x,t)$ be the solution of* (SC). *Denote, for $i \in \mathbb{N}$*

$$\phi_i(t) = \int_0^\infty x^i n(x,t)\,dx.$$

When these expressions are well defined, ϕ_0 is non-increasing, ϕ_1 is constant and ϕ_2 is increasing. Furthermore

$$\phi_0'(t) = -\frac{1}{2}\int_0^\infty \int_0^\infty K(x,y)n(x,t)n(y,t)\,dy\,dx. \tag{3.1}$$

These results can be found for example in Dubovskii ([Dub94]). We shall usually consider, for normalisation reasons

$$\phi_1(0) = \phi_1(t) = 1. \tag{3.2}$$

Let us denote

$$p(x,t) = xn(x,t), \quad x \geq 0, \quad t > 0. \tag{3.3}$$

By using (3.2), $(p(x,t))_x$ is the probability density of a positive random variable. We have the following characterisation:

PROPOSITION 3.2. *Let X_t be a positive random variable of density $p(x,t) = xn(x,t)$ and $\tilde{X}_t$ a random variable independent from X_t and with same distribution. Then $n(x,t)$ is solution of* (SC), *if and only if*

$$\frac{d}{dt}E(f(X_t)) = E\left(\frac{f(X_t+\tilde{X}_t) - f(X_t)}{\tilde{X}_t} K(X_t, \tilde{X}_t)\right), \tag{3.4}$$

for all smooth functions f.

REMARK 3.3. It suffices for example to consider f derivable, with compact support.

PROOF. Let us evaluate

$$\begin{aligned}\frac{d}{dt}E(f(X_t)) &= \int_0^\infty f(x)\frac{\partial p}{\partial t}(x,t)\,dx \\ &= \int_0^\infty \frac{x}{2} f(x) \int_0^x n(y,t)n(x-y,t)K(y,x-y)\,dy\,dx \\ &\quad - \int_0^\infty f(x)\,x\,n(x,t)\int_0^\infty K(x,y)n(y,t)\,dy\,dx.\end{aligned}$$

By using the variable change

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x-y \\ y \end{pmatrix}$$

in the first term on the right hand, we deduce

$$\begin{aligned}\frac{d}{dt}E(f(X_t)) &= \int_0^\infty \int_0^\infty \frac{(x+y)f(x+y)}{2} K(x,y)n(x,t)n(y,t)\,dy\,dx \\ &\quad - \int_0^\infty \int_0^\infty xf(x)K(x,y)n(x,t)n(y,t)\,dy\,dx \\ &= \int_0^\infty \int_0^\infty \left(\frac{f(x+y)-f(x)}{y}\right) K(x,y)p(x,t)p(y,t)\,dy\,dx.\end{aligned}$$

For the last equality we used the symmetry of K. This ends the proof of the proposition. □

3.2. – Continuous coagulation equation with constant kernel

For the constant kernel, a time normalisation is more interesting than the mass normalisation that we have announced in the general case. For this case we shall use this normalisation.

Let us first remark that the result in (1.15) applies in particular for constant and additive kernels, once the initial condition satisfies (C_0). We can deduce thus an existence and uniqueness result for the solution of (SC1) and (SC+). The equation (SC) becomes, for $K(x,y)=1$

$$\text{(SC1)} \qquad \begin{cases} \dfrac{\partial}{\partial t} n(x,t) = \dfrac{1}{2}\displaystyle\int_0^x n(y,t)n(x-y,t)\,dy - n(x,t)\int_0^\infty n(y,t)\,dy \\ n(x,0) = n_0(x). \end{cases}$$

For the same kernel, the differential equation (3.1), can be written

$$\phi_0'(t) = -\frac{\phi_0^2(t)}{2}. \tag{3.5}$$

We obtain

$$\phi_0(t) = \frac{2}{t+\alpha}$$

where $\alpha > 0$ is given by

$$\alpha = \frac{2}{\phi_0(0)}.$$

Let us denote by

$$p(x,t) = \frac{t+\alpha}{2} n(x,t), \tag{3.6}$$

such that $(p(x,t))_{x\geq 0}$ is the probability distribution of a positive random variable. There is a similar result to the Proposition 3.2 and it writes, for this particular case

PROPOSITION 3.4. *Let X_t denote a random variable with probability density $p(x,t)$ given by* (3.6)*, and $\tilde{X}_t$ an independent copy of X_t. Then, $n(x,t)$ is solution of* (SC1) *if and only if, for any smooth function f we have:*

$$\frac{d}{dt}E(f(X_t)) = \frac{1}{t+\alpha}(E(f(X_t+\tilde{X}_t)) - E(f(X_t))). \tag{3.7}$$

PROOF. We proceed as for the proof of the Proposition 3.2. Let us evaluate

$$\begin{aligned}
&\frac{d}{dt}E[f(X_t)] \\
&= \frac{1}{2}\int_0^\infty f(x)n(x,t)\,dx + \frac{t+\alpha}{2}\int_0^\infty \frac{f(x)}{2}\left(\int_0^x n(y,t)n(x-y,t)\,dy\right)dx \\
&\quad - \frac{t+\alpha}{2}\int_0^\infty f(x)n(x,t)\,dx \int_0^\infty n(y,t)\,dy \\
&= -\frac{1}{2}\int_0^\infty f(x)n(x,t)\,dx + \frac{t+\alpha}{4}\int_0^\infty\int_0^\infty f(x+y)n(y,t)n(x,t)\,dy\,dx \\
&= \frac{1}{t+\alpha}\int_0^\infty\int_0^\infty f(x+y)p(y,t)p(x,t)\,dy\,dx - \frac{1}{t+\alpha}\int_0^\infty f(x)p(x,t)\,dx\,.
\end{aligned}$$

This achieves the proof. □

This result allows us to describe entirely, for any initial condition, the solution of the Smoluchowski's coagulation equation with constant kernel, (SC1).

THEOREM 3.5. *Let T_t be a random variable of geometric law, with parameter $\frac{t}{t+\alpha}$ i.e.*

$$\forall k \geq 1,\ P(T_t = k) = \frac{\alpha}{t+\alpha}\left(\frac{t}{t+\alpha}\right)^{k-1}. \tag{3.8}$$

Let $(Y_i)_{i\geq 1}$ denote a sequence of i.i.d. random variables having same law as X_0, of probability density $p(x,0)$, and independent from T_t. Define

$$X_t = \sum_{i=1}^{T_t} Y_i,$$

and denote by $p(x,t)$ the distribution of the random variable X_t. Then

$$n(x,t) = \frac{2}{t+\alpha} p(x,t),$$

is the solution of the Smoluchowski's coagulation equation corresponding to the constant kernel (SC1).

PROOF. We prove this result by using Laplace transforms. Let us apply the equation (3.7) with $f_\lambda(x) = e^{-\lambda x}$. Denote by $\psi(\lambda,t) = E(e^{-\lambda X_t})$, the Laplace transform of X_t and by $g(\lambda) = \psi(\lambda,0)$, the Laplace transform of the initial condition. We deduce

$$\begin{cases} \dfrac{\partial}{\partial t}\psi(\lambda,t) = \dfrac{1}{t+\alpha}(\psi^2(\lambda,t) - \psi(\lambda,t)) \\ \psi(\lambda,0) = g(\lambda). \end{cases} \tag{3.9}$$

After integration we get

$$\frac{1}{(t+\alpha)\psi(\lambda,t)} = \frac{1}{t+\alpha} + d(\lambda), \tag{3.10}$$

where $d(\lambda)$ is a function depending only on λ. We obtain so the form of the Laplace transform

$$\psi(\lambda,t) = \frac{1}{(t+\alpha)d(\lambda)+1} \tag{3.11}$$

under the initial condition

$$\psi(\lambda,0) = g(\lambda) = \frac{1}{\alpha d(\lambda)+1}. \tag{3.12}$$

Let us also remark that the equation (3.10) insures that

$$(t+\alpha)\,d(\lambda)+1\neq 0.$$

From (3.12) we deduce the expression of d

$$d(\lambda)=\left(\frac{1}{g(\lambda)}-1\right)\frac{1}{\alpha}. \tag{3.13}$$

By reporting in (3.11) the form of d found in (3.13), we obtain for ψ the following formula:

$$\psi(\lambda,t)=\frac{\alpha g(\lambda)}{(t+\alpha)(1-g(\lambda)+\frac{\alpha}{t+\alpha}g(\lambda))}=\frac{\alpha g(\lambda)}{(t+\alpha)(1-\frac{t}{t+\alpha}g(\lambda))}. \tag{3.14}$$

As $g(\lambda)<1$, we can expand in integer series this expression and obtain

$$\psi(\lambda,t)=\frac{\alpha g(\lambda)}{t+\alpha}\sum_{k\geq 1}\left(\frac{t}{t+\alpha}\right)^{k-1}(g(\lambda))^{k-1}. \tag{3.15}$$

$g(\lambda)$ being a Laplace transform it is not difficult to verify that $\psi(\lambda,t)$ is also a Laplace transform (we can use the structure of ψ in terms of convolution). We have thus construct a solution of (3.9); more precisely, the equation (3.7) is satisfied for all exponential functions $f_\lambda(x)=e^{-\lambda x}$. This family is sufficiently large in order that (3.7) be verified by all smooth functions. This ends the proof of the Theorem 3.5, because the equation (3.7) is nothing else that the integral version of the Smoluchowski's coagulation equation (SC1). □

Let us focus now on the asymptotic behaviour of the random variables introduced in the Theorem 3.5.

THEOREM 3.6. *Consider the notations of Theorem 3.5. For any random variable X_0 with probability density $p(x,0)=\frac{\alpha}{2}n(x,0)$, we have, independently of the initial condition and for all fixed t*

$$a\,X_{\frac{t}{a}}\xrightarrow[a\to 0]{(d)}R_{\frac{t}{2}},$$

where R_t is the square of a two-dimensional Bessel process starting from the origin.

PROOF. We shall prove this convergence by using Laplace transforms. Let us remark first that, if the initial condition has a first order moment, we can write

$$g(\lambda)=E(e^{-\lambda X_0})=1-E(X_0)+o(\lambda).$$

We obtain, by using (3.14)

$$\begin{aligned}\psi(\lambda a,\frac{t}{a})&=\frac{\alpha g(\lambda a)}{\frac{t}{a}+\alpha-\frac{t}{a}g(\lambda a)}\\&=\frac{\alpha(1+o(1))}{\frac{t}{a}+\alpha-\frac{t}{a}(1-\lambda a\frac{\alpha}{2}+o(a))}\\&=\frac{1+o(1)}{1+\frac{\lambda t}{2}+o(1)}.\end{aligned}$$

By letting a goes to zero we obtain the result of the theorem. □

REMARK 3.7 R_t "corresponds" to the solution of (SC1) with initial condition a Dirac mass, δ_0. Consequently, for any initial condition, the scaled solution of the continuous Smoluchowski's coagulation equation, with constant kernel, converges to the solution of (SC1), with initial condition the Dirac mass.

By applying known results on the density of the square of a Bessel process, we deduce an exact solution for the Smoluchowski's coagulation equation (SC1)

$$n(x,t) = \frac{4}{t^2}\exp\left(-\frac{2x}{t}\right). \tag{3.16}$$

3.3. – Additive and multiplicative kernels for the continuous coagulation equation

We shall prove in this section that the solutions of the additive and multiplicative kernels are connected. We make first some remarks that simplify these equations. We will suppose in what follows, that

$$\phi_1(0) = \int_0^\infty xn(x,0)\,dx = 1. \tag{3.17}$$

In the sequel we will denote by $\overset{+}{n}(x,t)$ (respectively $\overset{*}{n}(x,t)$), a solution for the Smoluchowski's coagulation equation with additive kernel (respectively multiplicative).

REMARK 3.8. For the additive kernel $K(x,y) = x+y$, the equation (3.1) becomes

$$\phi_0'(t) = -\phi_0(t) \text{ and so } \phi_0(t) = \beta e^{-t},$$

where

$$\beta = \phi_0(0) = \int_0^\infty \overset{+}{n}(x,0)\,dx.$$

The equation (SC) can be written in this case

$$\text{(SC+)}\quad \begin{cases} \dfrac{\partial \overset{+}{n}}{\partial t}(x,t) &= \dfrac{x}{2}\displaystyle\int_0^x \overset{+}{n}(y,t)\overset{+}{n}(x-y,t)dy - x\overset{+}{n}(x,t)\int_0^\infty \overset{+}{n}(y,t)dy - \overset{+}{n}(x,t) \\ &= \dfrac{x}{2}\displaystyle\int_0^x \overset{+}{n}(y,t)\overset{+}{n}(x-y,t)\,dy - x\overset{+}{n}(x,t)\beta e^{-t} - \overset{+}{n}(x,t) \\ \overset{+}{n}(x,0) &= \overset{+}{n}_0(x). \end{cases}$$

Let us also write the equation (SC) for the multiplicative kernel.

$$\text{(SC}*\text{)}\quad \begin{cases} \dfrac{\partial \overset{*}{n}}{\partial t}(x,t) &= \dfrac{1}{2}\displaystyle\int_0^x y(x-y)\overset{*}{n}(y,t)\overset{*}{n}(x-y,t)dy - x\overset{*}{n}(x,t)\int_0^\infty y\overset{*}{n}(y,t)\,dy \\ &= \dfrac{1}{2}\displaystyle\int_0^x y(x-y)\overset{*}{n}(y,t)\overset{*}{n}(x-y,t)\,dy - x\overset{*}{n}(x,t) \\ \overset{*}{n}(x,0) &= \overset{*}{n}_0(x). \end{cases}$$

We can now express the connection between the solutions of the equations (SC+) and (SC$*$).

THEOREM 3.9. *Let* $\overset{+}{n}(x,t)$ *denote a solution of the Smoluchowski's coagulation equation with additive kernel* (SC+), *and initial condition* $\overset{+}{n}_0(x)$. *Then* $\overset{*}{n}(x,t)$ *is a solution for the equation with multiplicative kernel* (SC∗), *where we have denoted by*

$$\overset{*}{n}(x,t) = \frac{1}{T-t}\frac{1}{x}\overset{+}{n}\left(x, -\log\left(1-\frac{t}{T}\right)\right), \forall t < T, \tag{3.18}$$

and $T = \int_0^\infty \overset{+}{n}_0(y)\,dy$.

PROOF. We remark first that the initial condition (3.17), that we usually impose, is really satisfied for $\overset{*}{n}$. We have

$$\begin{aligned}\int_0^\infty x\overset{*}{n}(x,0)\,dx &= \frac{1}{T}\int_0^\infty \overset{+}{n}(x,0)\,dx\\ &= 1.\end{aligned}$$

We prove now that $\overset{*}{n}$ satisfies the Smoluchowski's equation (SC∗). We have

$$\begin{aligned}\frac{\partial}{\partial t}\overset{*}{n}(x,t) &= \frac{1}{(T-t)^2}\frac{1}{x}\overset{+}{n}\left(x,-\log\left(1-\frac{t}{T}\right)\right) + \frac{1}{(T-t)^2}\frac{1}{x}\frac{\partial}{\partial t}\overset{+}{n}\left(x,-\log\left(1-\frac{t}{T}\right)\right)\\ &= \frac{1}{(T-t)^2}\frac{1}{x}\overset{+}{n}\left(x,-\log\left(1-\frac{t}{T}\right)\right)\\ &\quad + \frac{1}{2}\frac{1}{(T-t)^2}\frac{1}{x}\int_0^x x\overset{+}{n}\left(y,-\log\left(1-\frac{t}{T}\right)\right)\overset{+}{n}\left(x-y,-\log\left(1-\frac{t}{T}\right)\right)dy\\ &\quad - \frac{1}{(T-t)^2}\frac{1}{x}\overset{+}{n}\left(x,-\log\left(1-\frac{t}{T}\right)\right)\\ &\quad - \frac{1}{(T-t)^2}\overset{+}{n}\left(x,-\log\left(1-\frac{t}{T}\right)\right)\int_0^\infty \overset{+}{n}\left(y,-\log\left(1-\frac{t}{T}\right)\right)dy,\end{aligned}$$

because $\overset{+}{n}$ is solution of (SC+). We conclude that

$$\frac{\partial}{\partial t}\overset{*}{n}(x,t) = \frac{1}{2}\int_0^x y(x-y)\overset{*}{n}(y,t)\overset{*}{n}(x-y,t)\,dy - x\overset{*}{n}(x,t).$$

Here we have used that for the additive solution

$$\int_0^\infty \overset{+}{n}(y,t)\,dy = Te^{-t} \text{ with } T = \int_0^\infty \overset{+}{n}(x,0)\,dx.$$

This ends the proof of the Theorem 3.9. □

REMARK 3.10. The transformation in Theorem 3.9 emphasises a finite time existence interval for the solution in the multiplicative case. We find thus already known results for this kernel ($T_{\mathrm{gel}} < \infty$, Norris ([Nor99]))

We have also the inverse transformation.

THEOREM 3.11. *If $\overset{*}{n}(x,t)$ is a solution of the Smoluchowski's coagulation equation with multiplicative kernel* (SC$*$) *and initial condition $\overset{*}{n}_0(x)$, then $\overset{+}{n}(x,t)$ is a solution of the Smoluchowski's equation with additive kernel* (SC$+$), *where we have denoted by:*

$$\overset{+}{n}(x,t) = x\,T\,e^{-t}\overset{*}{n}(x, T(1-e^{-t}))$$

and $T = \left(\int_0^\infty x^2\,\overset{*}{n}_0(x)\,dx\right)^{-1}$.

PROOF. The proof of this theorem is similar to the one of the Theorem 3.9. The choice of T insures that (3.17) is satisfied, i.e.

$$\int_0^\infty x\overset{+}{n}(x,0) = 1. \qquad \square$$

REMARK 3.12. These transformations (Theorems 3.9 and 3.11), apply also to the discrete case. It suffices to replace integrals by sums.

3.3.1. – Existence of Solutions for (SC$+$) and (SC$*$)

The aim of this part is to prove existence of solutions for (SC$+$) and (SC$*$) by employing probabilistic methods and the preceding transformations. We shall first recall an existence result which can be found for example in Dubovskii ([Dub94]). Consider the equation (SC$*$). We have

THEOREM 3.13. *For any initial condition $\overset{*}{n}(x,0)$, $x \geq 0$ which satisfies $\int_0^\infty x\overset{*}{n}(x,0)\,dx = 1$ and $0 < \int_0^\infty x^2\overset{*}{n}(x,0)\,dx < \infty$, there exists an unique solution of* (SC$*$) *defined for $t \in [0,T)$ where*

$$T = \left(\int_0^\infty x^2\overset{*}{n}(x,0)\,dx\right)^{-1}.$$

PROOF. The aim of what follows is to prove this theorem by using probabilistic techniques. In order to obtain this result, we apply the Proposition 3.2 on this particular case. We deduce that

$$\frac{d}{dt}E(f(X_t)) = E((f(X_t+\tilde{X}_t) - f(X_t))X_t), \tag{3.19}$$

where X_t has density $x\overset{*}{n}(x,t)$ and $\tilde{X}_t$ is an independent copy of the random variable X_t. We apply the equation (3.19) to functions of the form $f_\lambda(x) = e^{-\lambda x}$.

Denote by $\psi(\lambda, t) = E\left(e^{-\lambda X_t}\right)$ the Laplace transform of X_t. ψ satisfies the non linear hyperbolic partial differential equation:

$$\left\{\begin{array}{l} \dfrac{\partial \psi}{\partial t} = (1-\psi)\dfrac{\partial \psi}{\partial \lambda} \\ \psi(\lambda, 0) = g(\lambda). \end{array}\right. \tag{3.20}$$

We will construct a solution for this equation under the initial condition

$$\psi(\lambda, 0) = \int_0^\infty e^{-\lambda x} x \overset{*}{n}(x, 0)\, dx$$

and prove that it conserves, for all t, the property of being a Laplace transform once that $g(\lambda) = \psi(\lambda, 0)$ is a Laplace transform.

Let us denote, for $t \in \left[0, -\frac{1}{g'(0)}\right[$

$$\psi(\lambda_0 + (g(\lambda_0) - 1)t, t) = g(\lambda_0). \tag{3.21}$$

PROPOSITION 3.14. *The function ψ defined by* (3.21) *is a solution of the partial differential equation* (3.20).

PROOF. This proposition doesn't really need a proof. By construction the result is true. Indeed, we constructed this solution by using the level sets of the equation (3.20). □

We want to prove that $\psi(\lambda, t)$, solution of (3.20), is a Laplace transform. We shall use the Karamata theorem (cf. ([Fel66]), p. 439). In order to apply it we need some auxiliary results (Lemmas 3.15 and 3.16).

LEMMA 3.15. *The function ψ, solution of* (3.20)*, is completely monotone, that is*

$$\forall k \geq 0,\ (-1)^k \frac{\partial^k \psi}{\partial \lambda^k} \geq 0. \tag{3.22}$$

PROOF. The proof will be done by recurrence on k: the result is true for $k = 0$, by construction of ψ.

Let us denote by $f_k(t) = \frac{\partial^k \psi}{\partial \lambda^k}(\lambda(t), t)$ where $\lambda(t) = \lambda_0 + (g(\lambda_0) - 1)t$.

We suppose that:

$$\forall j \in [\![1, k-1]\!],\ (-1)^j f_j(t) \geq 0. \tag{3.23}$$

We get then:

$$f_k'(t) = \lambda'(t) \frac{\partial^{k+1}}{\partial \lambda^{k+1}} \psi(\lambda(t), t) + \frac{\partial^{k+1}}{\partial \lambda^k \partial t} \psi(\lambda(t), t). \tag{3.24}$$

By taking the derivative in the equation (3.20), k times with respect to λ, we obtain

$$\begin{aligned}\frac{\partial^{k+1}\psi}{\partial\lambda^k\,\partial t}(\lambda,t)&=(1-\psi(\lambda,t))\frac{\partial^{k+1}}{\partial\lambda^{k+1}}\psi(\lambda,t)\\&\quad-\sum_{j=1}^{k}\binom{k}{j}\frac{\partial^j}{\partial\lambda^j}\psi(\lambda,t)\frac{\partial^{k-j+1}}{\partial\lambda^{k-j+1}}\psi(\lambda,t).\end{aligned}\tag{3.25}$$

This gives, by using (3.24) and recalling that we are on a level set $(1-\psi(\lambda(t),t)+\lambda'(t)=0)$

$$\begin{aligned}f_k'(t)&=-(k+1)\frac{\partial}{\partial\lambda}\psi(\lambda(t),t)f_k(t)\\&\quad-\sum_{j=2}^{k-1}\binom{k}{j}\frac{\partial^j}{\partial\lambda^j}\psi(\lambda(t),t)\frac{\partial^{k-j+1}}{\partial\lambda^{k-j+1}}\psi(\lambda(t),t)\\&=-(k+1)f_1(t)f_k(t)-\sum_{j=2}^{k-1}\binom{k}{j}f_j(t)f_{k+1-j}(t).\end{aligned}\tag{3.26}$$

We remark now that each term in the previous sum has same sign as $(-1)^{k+1}$, by the recurrence hypothesis (3.23). In conclusion, we have a first order partial differential equation, that is:

$$u'(t)=a(t)u(t)+b(t)\tag{3.27}$$

with $u(0)$ and $b(t)$ of same sign.

In this case, the sign of $u(t)$ is conserved, for all t in the definition domain of the function u, solution of the equation (3.27). □

Furthermore:

LEMMA 3.16. *For all* $t\in\left[0,-\frac{1}{g'(0)}\right[$, *we have*

$$\lim_{\lambda\to 0}\psi(\lambda,t)=1.$$

PROOF. We calculate the ordinate of the intersection between the level set and the axis $\lambda=0$. Call $t_0(\lambda_0)$ the intersection point of the level set passing through the point $(\lambda_0,0)$, with this axis. We have

$$t_0(\lambda_0)=\frac{\lambda_0}{1-g(\lambda_0)}.$$

By taking derivatives in this expression we get

$$t_0'(\lambda_0)=\frac{1-g(\lambda_0)-\lambda_0 g'(\lambda_0)}{(1-g(\lambda_0))^2}.$$

This last expression is obviously positive ($g(\lambda)$ is a Laplace transform). Thus $t_0(\lambda_0)$ is increasing (see figure 1) and equals $-\frac{1}{g'(0)}$ in 0. We deduce that, for all $t \in \left[0, -\frac{1}{g'(0)}\right[$,

$$\lim_{\lambda \to 0} \psi(\lambda, t) = 1. \qquad \square$$

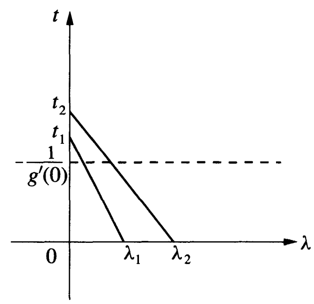

Fig. 1. Behaviour of $t_0(\lambda_0)$.

The results of Lemmas 3.15 and 3.16 prove (by the Karamata theorem, ([Fel66]), p. 439) the following proposition:

PROPOSITION 3.17. *For all* $t \in \left[0, -\frac{1}{g'(0)}\right[$, ψ *is the Laplace transform of a probability density.*

By using the resolution of the partial differential equation satisfied by the Laplace transform, we have proved that the equation (3.19) is true for all exponential functions $f_\lambda(x) = e^{-\lambda x}$. This family forms a large family of functions and the equation will be satisfied by all smooth functions. On the other hand this equation is the integral form of the Smoluchowski's coagulation equation (SC$*$). This ends the proof of the Theorem 3.13. $\square$

REMARK 3.18 Similar techniques, based on Laplace transform, have been used by Ernst, Ziff and Hendricks ([EZH84]) for the multiplicative kernel, in order to find the expression of $\int_0^\infty x\overset{*}{n}(x,t)dx$ beyond the gelification time.

Let us note now the consequence of Theorems 3.13 and 3.11.

COROLLARY 3.19. *For any initial condition* $\overset{+}{n}(x, 0)$, $x \geq 0$ *satisfying*

$$\int_0^\infty x\overset{+}{n}(x, 0)\, dx = 1,$$

there exists an unique solution of (SC+) *defined for all* $t \in [0, +\infty[$.

3.3.2. – Convergence of solutions

In the last part we state the renormalisation theorems. More precisely, we shall see now that the solutions of (SC+) have the same asymptotic behaviour under initial conditions not too restrictive. Let us state this theorem:

THEOREM 3.20. *For $\overset{+}{n}(x,t)$ solution of the Smoluchowski's coagulation equation with additive kernel* (SC+), *let us denote $\overset{+}{p}(x,t) = x\,\overset{+}{n}(x,t)$. We suppose that there exists a constant $C > 0$ such that $m_k(0) = \int_0^\infty x^k \overset{+}{p}(x,0)\,dx \le C^k \frac{(2k)!}{k!}$. We have then, for fixed t:*

$$a^2 X_{\log \frac{t}{a}} \xrightarrow[a\to 0]{(\mathrm{d})} R_{A_1 t^2}, \tag{3.28}$$

where R_t denotes the square of the one-dimensional Bessel process started at the origin and $A_1 = m_1(0) = \int_0^\infty x\overset{+}{p}(x,0)\,dx = \int_0^\infty x^2\overset{+}{n}(x,0)\,dx$.

REMARK 3.21. The condition of this theorem on the initial condition is satisfied by all random variables whose moments are dominated by the moments of the square of a Gaussian random variable. This is true for example for random variables having compact support.

PROOF. In order to make the proof we need some auxiliary results and remarks. Proposition 3.2 gives in this particular case, the following result: for X_t a random variable with probability density $p(x,t)$ and f a smooth function, we have

$$\frac{d}{dt}E(f(X_t)) = E\left(\frac{X_t + \tilde{X}_t}{\tilde{X}_t}(f(X_t + \tilde{X}_t) - f(X_t))\right), \tag{3.29}$$

where $\tilde{X}_t$ denotes an independent copy of the random variable X_t.

To get existence we used functions f of the form $e^{-\lambda x}$. Another way to treat the problem is to find the moments of X_t, by using a recurrence formula.

Let us apply formula (3.29) for power functions $f_k(x) = x^k$. Denote by

$$m_k(t) = \int_0^\infty x^k \overset{+}{p}(x,t)\,dx. \tag{3.30}$$

For $f_1(x) = x$, (3.29) writes

$$m_1'(t) = 2m_1(t),$$

that is

$$m_1(t) = A_1 e^{2t} \text{ where } A_1 = m_1(0).$$

Furthermore, by using (3.29) for $f_k(x) = x^k$, we obtain

$$m_k'(t) = (k+1)m_k(t) + \sum_{j=1}^{k-1}\binom{k+1}{j} m_j(t)m_{k-j}(t). \tag{3.31}$$

Simple computation for first order moments gives

$$\begin{aligned} m_1(t) &= A_1 e^{2t}, & A_1 &= m_1(0), \\ m_2(t) &= 3{A_1}^2 e^{4t} + A_2 e^{3t}, & A_2 &= m_2(0) - 3A_1^2, \\ m_3(t) &= 15{A_1}^3 e^{6t} + 10A_1A_2e^{5t} + A_3e^{4t}, & A_3 &= m_3(0) - 15A_1^3 - 10A_1A_2. \end{aligned}$$

These results and formula (3.31) allow us to obtain the general form of m_k.

LEMMA 3.22. *For all integer k, $m_k(\log t)$ is a polynomial on t of degree $2k$, with valuation greater or equal to $k+1$ and dominating coefficient: $\frac{(2k)!}{2^k k!}A_1^k$, where $A_1 = m_1(0)$.*

PROOF. We have already test this result for $k = 1$ (and even for $k = 0$). Suppose it valid until $k-1$ and solve the equation (3.31) by using the method of constant variation. We obtain

$$m_k(t) = e^{(k+1)t}\Gamma(t), \tag{3.32}$$

where $\Gamma(t)$ is such that:

$$e^{(k+1)t}\Gamma'(t) = \sum_{j=1}^{k-1} \binom{k+1}{j} m_j(t) m_{k-j}(t). \tag{3.33}$$

By using the recurrence hypothesis, $\Gamma'(\log t)$ is a polynomial of degree $k-1$ and dominating coefficient $\frac{A_1^k}{2^k}\sum_{j=1}^{k-1}\binom{k+1}{j}\frac{(2j)!(2k-2j)!}{j!(k-j)!}$.

We can prove also the following result (see Lemma 4.1, given in the appendix):

$$\sum_{j=1}^{k-1}\binom{k+1}{j}\frac{(2j)!(2k-2j)!}{j!(k-j)!} = (k-1)\frac{(2k)!}{k!}.$$

We deduce that $\Gamma(\log t)$ is a polynomial on t of degree $k-1$ and dominating coefficient $\frac{(2k)!}{2^k k!}A_1^k$. This ends the proof. □

Let us write down the Laplace transform of the random variable X_t.

$$\psi(\lambda, t) = \sum_{k\geq 0} \frac{(-1)^k \lambda^k}{k!} m_k(t). \tag{3.34}$$

Evaluate this function at $(\lambda a^2, \log \frac{t}{a})$.

$$\psi(\lambda a^2, \log \frac{t}{a}) = \sum_{k\geq 0} \frac{(-1)^k\lambda^k a^{2k}}{k!} m_k(\log \frac{t}{a}). \tag{3.35}$$

By using the result in Lemma 3.22 we can remark that, each term of this series satisfies

$$\lim_{a\to 0}\left((\lambda a^2)^k m_k(\log \frac{t}{a})\right) = A_1^k \frac{(2k)!}{2^k k!}\lambda^k t^{2k}. \tag{3.36}$$

We shall prove that we can consider the limit as a goes to 0 in the sum of (3.35). In order to obtain this convergence, we need the following result:

LEMMA 3.23. *For any initial condition of the Smoluchowski's coagulation equation* (SC+), *such that* $m_k(0) \leq C^k \frac{(2k)!}{k!}$ *for fixed C and for all k, we have*

$$m_k(t) \leq C^k \frac{(2k)!}{k!} e^{2kt}, \quad \forall t \geq 0.$$

PROOF. We shall use the same method as the one used in order to obtain the form of m_k. We get from (3.32) and (3.33)

$$\Gamma'(t) \leq \frac{C^k}{k-1} \frac{(2k)!}{k!} e^{(k-1)t}. \tag{3.37}$$

We have also

$$\Gamma(t) = \Gamma(0) + \int_0^t \Gamma'(s) ds,$$

which yields

$$\Gamma(t) \leq C^k \frac{(2k)!}{k!} e^{(k-1)t} + \Gamma(0) - \frac{(2k)!}{k!} C^k,$$

and $\Gamma(0) = m_k(0)$. The result is then proved. □

Lemma 3.23 insures the normal convergence, with respect to a, of the series appearing in the expression of $\psi(\lambda a^2, \log \frac{t}{a})$ (see (3.35)), as soon as $\frac{4\lambda t^2}{C} \leq 1$. We can now conclude the proof of the Theorem 3.20.

By using the previous normal convergence, we easily deduce

$$\lim_{a \to 0} \psi(\lambda a^2, \log \frac{t}{a}) = \sum_{k \geq 0} \frac{(2k)!}{(k!)^2} \frac{(-1)^k A_1^k \lambda^k t^{2k}}{2^k} = \frac{1}{\sqrt{1 + 2A_1 \lambda t^2}}. \tag{3.38}$$

This last limit is also the Laplace transform of the square of a one-dimensional Bessel process starting from the origin. □

As in the previous section, this convergence allows us to find a particular solution of (SC+):

$$n_1(x,t) = \frac{1}{\sqrt{2\pi}} e^{-t} x^{-\frac{3}{2}} e^{-\frac{x}{2} e^{-2t}}.$$

Other solutions can be obtained by scaling, more precisely, consider

$$n_{A_1}(x,t) = \frac{1}{A_1^2} n_1 \left(\frac{x}{A_1}, t \right)$$

so

$$n_{A_1}(x,t) = \frac{1}{\sqrt{A_1}} \frac{1}{\sqrt{2\pi}} e^{-t} x^{-\frac{3}{2}} e^{-\frac{x}{2A_1} e^{-2t}}$$

is a solution of (SC+).

Let us also express the similar result for the solutions of (SC∗). By using Theorem 3.9 we have

THEOREM 3.24. *Let $\overset{*}{n}(x,t)$ be a solution of the Smoluchowski's coagulation equation with multiplicative kernel* (SC$*$). *Let*

$$\overset{*}{p}(x,t) = x^2\,(T-t)\overset{*}{n}(x,t),$$

where

$$T = \left(\int_0^\infty x^2\overset{*}{n}(x,0)\,dx\right)^{-1}.$$

Denote by Y_t a random variable with probability density $\overset{*}{p}(x,t)$. We suppose also that $m_k(0) = \int_0^\infty x^k\overset{*}{p}(x,0)\,dx \le C^k\frac{(2k)!}{k!}$.

Then, by fixing t, we have:

$$a^2 Y_{T(1-\frac{t}{a})} \xrightarrow[a\to 0]{(\mathrm{d})} R_{A_1 t^2}, \tag{3.39}$$

where

$$A_1 = \int_0^\infty x\overset{*}{p}(x,0)dx = T\int_0^\infty x^3\overset{*}{n}(x,0)\,dx$$

and R_t is the square of a one-dimensional Bessel process, starting at the origin.

We conclude our study by making two remarks:

REMARK 3.25. We mention that Van Dongen and Ernst ([vDE85]) analysed the long time behaviour in the discrete case. Part of their approach is rather intuitive. See also the section 2.4 of Aldous' paper ([Ald99]) for a presentation on self similarity results for the solution of Smoluchowski's coagulation equation. We emphasise that our normalisation theorems prove that, any given solution converges, after employing a good scaling, to a self-similar solution of the equation. At our knowledge these results, stated on the previous forms, are new.

REMARK 3.26. The preceding normalisation theorems are true for fix t. It could be interesting and probably very difficult to obtain them in terms of processes.

Let us write in the following schema the solutions we got for the continuous case (by using Laplace transforms):

$K(x,y)$	$n(x,t)$	t
1	$\frac{4}{t^2}\exp\left(\frac{-2x}{t}\right)$	$0 < t < \infty$
$x+y$	$\frac{1}{\sqrt{2\pi}}e^{-t}\exp\left(-e^{-2t}\frac{x}{2}\right)\frac{1}{x^{\frac{3}{2}}}$	$-\infty < t < \infty$
xy	$\frac{1}{\sqrt{2\pi}}\frac{1}{x^{\frac{5}{2}}}\exp\left(-\frac{t^2 x}{2}\right)$	$-\infty < t < 0$

4. – Appendix

We shall prove now the result we used in the proof of Lemma 3.22.

LEMMA 4.1. *For any integer $k \geq 2$, we have the following identity:*

$$\sum_{j=1}^{k-1} \binom{k+1}{j} \frac{(2j)!(2k-2j)!}{j!(k-j)!} = (k-1)\frac{(2k)!}{k!}. \tag{4.1}$$

PROOF. This result can be obtained by recurrence. We present here a probabilistic proof, communicated by P. Chassaing.

The equality (4.1) can be written in the equivalent form

$$\binom{2k+1}{k} = \sum_{j=0}^{k} \binom{2j}{j} \frac{(2k-2j)!}{(k-j)!(k+1-j)!}. \tag{4.2}$$

Let us consider equiprobable paths $\omega = (S_0(\omega), \cdots, S_{2k+1}(\omega))$ such that $S_0 = 0$, $S_{n+1} = S_n \pm 1$. There are $\binom{2k+1}{k}$ such paths of length $2k+1$ having $S_{2k+1}(\omega) = 1$ (k descents in $2k+1$ steps). For each path ω such that $S_{2k+1}(\omega) = 1$, we note $T = \max\{0 \leq n \leq 2k / S_n(\omega) = 0\}$ the last instant where the path is 0 (this place exists and T is even). The number of paths for which $T = 2j$, equals the product of the number of paths which go from 0 to 0 in $2j$ steps (they are $\binom{2j}{j}$), with the number of paths having length $2k - 2j + 1$ which go from 0 to 1, without any 0 value (they are as much as Bernoulli excursions of length $2(k - j + 1)$, i.e. C_{k-j} where C_n denotes the n^{th} Catalan's number).

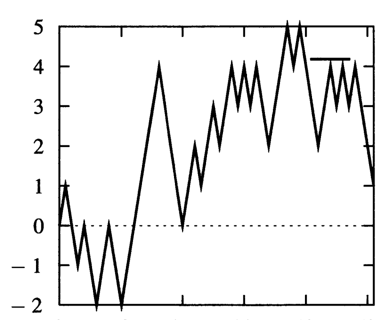

Fig. 2. Path description.

Let us recall the value for the Catalan's number:

$$C_n = \frac{(2n)!}{n!(n+1)!}.$$

Doing the sum over all possible values of j yields the identity (4.2). This ends the proof of the lemma. □

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